Importance of Conic Section “size” in the Derivation of Propositions X-XVI in Newton’s *Principia* Book I

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Abstract: The “direct problem” of the inverse-square law is examined, by retracing the derivations of Propositions X-XVI in Newton’s Principia. It is found that transfers of constants of proportionality are inconsistent between the propositions, leading to errors in the derived relations for multiple orbits. Proposition X has to be corrected for the relations of time period to the force law by including the area constants for conic sections. Propositions XI-XIII assume the expression $2h^2/L$ to be constant, that in turn implicitly assume a modified version of Kepler’s third law, while Propositions XIV-XVI apply this to multiple orbits leading to further “size” inconsistencies with the original form of Kepler’s third law. It has been shown how this relevance of “size” has been missed in the Newtonian literature, leading to a necessity of revising the existence of an exact inverse-square law.

Introduction

Newton’s force laws including the inverse-square law have received a thoroughgoing treatment not only in the 18th and 19th century, but also recently. In the past thirty years, the issue of the rigor of the Newtonian proof for the inverse argument of the inverse-square law – that the presence of an inverse-square law implies the existence of orbits which are conic sections – has received renewed attention. In addition to the arguments raised by Weinstock [1] [2] regarding the validity of this proof in the *Principia*, a series of papers – such as [3] [4] [5] [6] and [7] – have clarified the details of Newton’s proof and justified the rigor of the derivation of conics from the inverse-square law. They have also shown that the required proof not only was possible for Newton, but was most likely accomplished by him using the integral tables available to him at the time [3].

Although there has been extensive commentary on the functional dependences of the laws, such as the proportionality between the force F and the radial dependency $1/r^2$, the author is not aware of any recent detailed work on the corresponding constants of proportionality for these laws. This has been understandable since the original *Principia* was written in the language of geometric proportionality, and the later derivations using modern methods of calculus such as [8] utilize constants without a full examination of their interrelationships between propositions (described further in Sections 1-5). Therefore, the analysis in this paper proceeds from a different foundation: that of the use of the *constants of proportionality* or – simplistically called “sizes” in this work. Prior to an analysis of the propositions related directly to the inverse-square law, viz.

Propositions XI – XVI (Book I), Proposition X will be examined in section 1. Following this, the same concepts are applied for the remaining propositions sequentially.

1. Proposition X, Book I

In this proposition, the purpose is, according to Newton [9]:

> ***Prop. X. Prob. V.*** *Let a body revolve in an ellipse; it is required to find the law of the centripetal force tending toward the center of the ellipse.*

The relevant diagram is included from the *Principia* below.

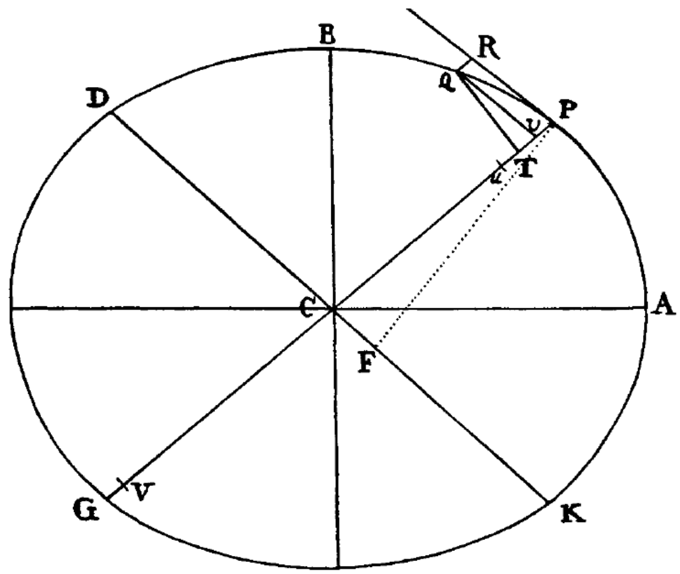

Fig. 1: Proposition X, with C as center of forces, and particle at P

It must be noted that the word "force" as used by Newton in these sections of the *Principia* refers to the force of *acceleration*, and therefore have dimensions of acceleration. The result of the proposition is to deduce that:

> Therefore (by prop. 6, corol. 5), the centripetal force is as $\frac{2\mathrm{BC}^2\times\mathrm{CA}^2}{\mathrm{PC}}$ inversely, that is (because $2\mathrm{BC}^2 \times \mathrm{CA}^2$ is given), as $\frac{1}{\mathrm{PC}}$ inversely, that is, as the distance PC directly.

For convenience of notation, the following variables are used:

$$a = \mathrm{CA};\ b = \mathrm{BC};\ r = \mathrm{PC};\ \mathrm{F} = \text{centripetal force}$$

Therefore, the proposition states that:

$$\mathrm{F} \propto \frac{\mathrm{PC}}{2\mathrm{BC}^2 \times \mathrm{CA}^2} = \frac{r}{2a^2b^2} \propto r \tag{1}$$

At this point it is important to note that the last proportionality (w.r.t 'r') only holds when the same ellipse is involved, i.e. where a and b can be considered constants. Where this no longer holds, e.g. between different ellipses, the proportionality of the force may differ. However, Corollary 2, Proposition X states that:

> COROLLARY 2. And the periodic times of the revolutions made in all ellipses universally around the same center will be equal. For in similar ellipses those times are equal (by prop. 4, corols. 3 and 8), while in ellipses having a common major axis they are to one another as the total areas of the ellipses directly and the particles of the areas described in the same time inversely; that is, as the minor axes directly and the velocities of bodies in their principal vertices inversely; that is, as those minor axes directly and the ordinates to the same point of the common axis inversely; and therefore (because of the equality of the direct and inverse ratios) in the ratio of equality.

Here, Proposition IV is invoked to justify the equality of times for ellipses. However, in Proposition IV Corollary 3, it is explained *only* for circles that:

$$t_1 = t_2 \leftrightarrow \mathrm{F} \propto r \tag{2}$$

where $t_1$and t_2 are the time periods of two circular orbits. In Corollary 8, this relationship is extended to "any similar figures whatever," with the caveat that the relationships are not directly, but only *ultimately* true (see [10], pg. 173). Newton highlights this disclaimer by mentioning that one must apply "uniform descriptions of areas for uniform motion," which is a natural result of taking the limit of a curve as being a circle. Any elliptical orbit can be approximated by a circular orbit at the limit, but the converse *does not hold*. Since it does not follow that a curve where the limit can be approximated by a circle (with the restrictions applied by the scholium to Lemma 11) can have the same relationships between parameters as that of the circle itself, equation (2) does not automatically hold for the ellipse, and will have to be derived from Proposition X itself.

Starting once more with equation (1):

$$\mathrm{F} \propto \frac{r}{2a^2b^2} \tag{1a}$$

Here, the denominator is related to the square of the area, since the area of an ellipse is πab. By substituting the areal velocity (conventionally taken to be $h/2$) and the time period (T) conjointly for the area, one gets:

$$\mathrm{F} \propto \frac{r}{2a^2b^2} = \frac{r}{2(h/2\pi)^2T^2} \propto \frac{r}{h^2T^2} \tag{3}$$

Where equal time periods are compared ($T_1 = T_2$), as in Proposition X Corollary 2, the relationship between two ellipses becomes:

$$\frac{F_1}{F_2} = \frac{r_1}{r_2} * \frac{h_2^2}{h_1^2} \neq \frac{r_1}{r_2} \ (unless\ h_1 = h_2) \tag{4}$$

Since for the areal velocities, there is no prior reason to assume that they are equal, Newton's statement that "*… For in similar ellipses those times are equal (by prop. 4, corols. 3 and 8)…*" is no longer valid. From equation (4), when time periods are equal, force is proportional to the distance if and only if the areal velocities of two ellipses are equal, and not otherwise. Hence conversely, the times will be equal only under the same condition. It is therefore incorrect to represent the proportionality of equation (1a) as that between the force and distance alone. Corollary 3 of Proposition IV is not applicable directly to ellipses and Corollary 8, applying only at the limit, does not support the proportionality either.

As to the latter part of Corollary 2, where the major axes are equal, the following figure (Fig. 2) will be used for reference. Here b_1 and b_2 are the minor axes of the two ellipses with the major axis a being common to both of them. The velocities at the principal vertices are v_1 and v_2 in the larger and the smaller orbit respectively, here taken in opposite directions for ease of representation (note that the ratios depend on the magnitudes, not the directions.)

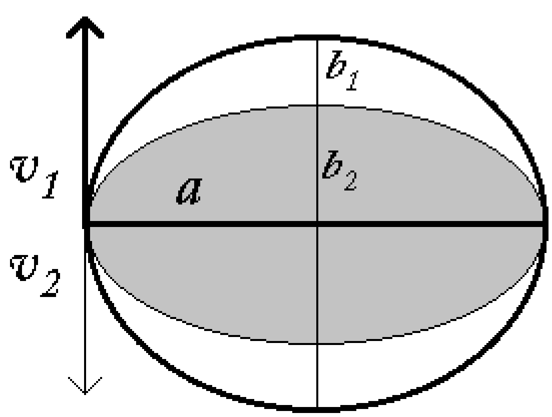

Fig. 2: Two ellipses, with common major axis "a" and minor axes $b_1 > b_2$.

According to the definition of areal velocity, the relation between time periods is given by:

$$\frac{T_1}{T_2} = \frac{A_1}{A_2} * \frac{dA_2/dt}{dA_1/dt} = \frac{\pi a b_1}{\pi a b_2} * \frac{a v_2}{a v_1} = \frac{b_1 v_2}{b_2 v_1} \tag{5}$$

Here A_i is the area of the i-th ellipse, and dA_i/dt is its rate of change. From this point onwards, Corollary 2 claims that the ratio in equation (4) is one of equality, by comparing ordinates at the limit. However, there is a straightforward way of proceeding, described as follows. Since the radius of curvature R of an ellipse at the major axis vertex is given by b^2/a, for the two ellipses, the following relation holds:

$$\frac{R_1}{R_2} = \frac{b_1^2/a}{b_2^2/a} = \left(\frac{b_1}{b_2}\right)^2 \tag{6}$$

From Proposition IV ($F \propto v^2/R$), and combining equations (6) and (1a) at the principal vertices on major axis:

$$\frac{F_1}{F_2} = \frac{v_1^2/R_1}{v_2^2/R_2} = \left(\frac{v_1}{v_2}\right)^2 * \frac{R_2}{R_1} = \left(\frac{b_2 v_1}{b_1 v_2}\right)^2 \tag{7a}$$

$$\frac{F_1}{F_2} = \frac{a}{a^2 {b_1}^2} \Big/ \frac{a}{a^2 {b_2}^2} = \left(\frac{b_2}{b_1}\right)^2 \tag{7b}$$

Therefore from equation (7a), (7b) and (5), it is straightforward to obtain:

$$\frac{T_1}{T_2} = \frac{b_1 v_2}{b_2 v_1} = \frac{b_1}{b_2} = \sqrt{\frac{F_2}{F_1}} \tag{8}$$

Hence, the velocities at the principal vertices are equal *but the time periods are not, contrary to what is deduced in Corollary 2 of Proposition X*. Time periods vary as the minor axes. Since the derivation just described does not depend on any other assumptions with regard to the force law and all the constants are accounted for, it does not suffer from the shortcomings of assuming a dependence of the force solely on the distance. Following up with the constants throughout the derivation is hence seen to be of utmost importance. It is quite surprising that in common references, such as [8] (pg. 90, (b) (i)) and [10] (pg. 218-224), this dependence of force on the constants of the ellipse has been completely neglected. In fact, the question is posed to the general student audience in [10] (pg. 216) that:

> This corollary claims that, no matter how the sides of the ellipse are squished in or stretched out, the periodic times will remain the same. Does this seem surprising to you? Would you expect a body to take as long to go around a little bitty orbit as around a huge one? Are you curious how Newton would prove such a thing? Can we see how the force law of I.10 would have that result?

Although the question is posed, the answer is not pursued systematically. Since the physical intuition is evidently surprised by the lack of effect of such a "size" difference, that relation has been followed up, and corrected for the neglect of this "size" (which is – in this particular case – $a^2 b^2$.)

2. Proposition XI, XII, XIII (Book I)

In these propositions, the force law for an object moving in a conic section is described. Since the expression from Proposition VI corollary 1 is used in these proofs, it is important to clarify the constants in it. The relevant diagram from *Principia* is reproduced below for clarity.

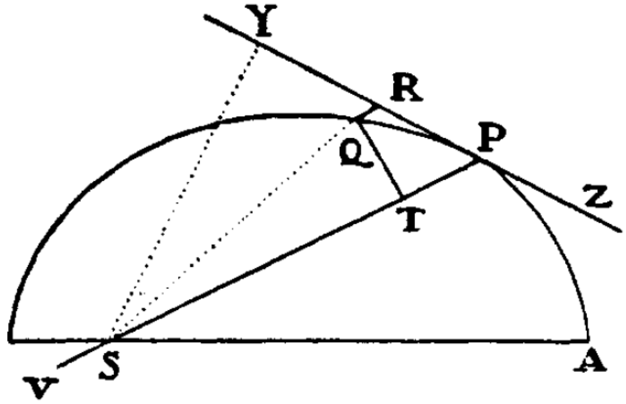

Fig. 3: Movement of an object at P on a curve, Proposition VI, Corollary 1, Book I

The expression for the force is ([8], pg. 77):

$$F \propto \frac{QR}{SP^2 QT^2} \ or \ F = \lim_{Q \to P} \frac{2h^2 QR}{SP^2 QT^2} \tag{9}$$

Where $h/2$ is the areal velocity, as usual. In the case of elliptic motion (Proposition XI), hyperbolic motion (Proposition XII), and parabolic motion (Proposition XIII) the following expression is obtained:

$$F = \frac{2h^2}{L \cdot SP^2} \tag{10}$$

Here, L is the principal latus rectum of the conic section in question, which is true at the limit of equation (9). The last line of each of the propositions ends with:

$$F \propto \frac{1}{SP^2} \tag{11}$$

In the transition from equation (10) to (11), it is assumed that $2h^2/L$ is a constant of proportion. While this is true as long as the treatment is restricted to *one* conic section, there is no necessity that it remains a constant when comparing *different* orbits and *different* bodies. The constant h representing areal velocity is related to the radius vector and tangential velocity at a point, and is independent of the latus rectum. If the two arbitrary constants, h and L, are to have a constant relation between them, that relationship has to be *proved* from the dynamics of the particle itself when applied to different orbits. An additional consequence of treating h^2/L as a constant can be demonstrated by the example of its use in an ellipse with major axis a, minor axis b and time period T:

$$\frac{2h^2}{L} = \frac{h^2}{a(1-e^2)} = \frac{2(2\pi ab/T)^2}{2b^2/a} = 4\pi^2 \frac{a^3}{T^2} \tag{12}$$

Assuming $2h^2/L$ to be a constant is *exactly* the same as assuming the validity of Kepler's third law, for a mean distance of a except that it stands for the major axis here (for more on this concept, see Section 4). The force F, as a function F (*h, L,* SP) can be reduced to F (SP) only under this condition. A physical or mathematical interpretation for the orbit requires a demonstration that $2h^2/L$ is necessarily a constant, *prior* to moving ahead with the next proposition that utilizes F as a function of SP alone. Hence the important fact to notice is that $2h^2/L$ is NOT proved to be a constant when comparing different orbits.

3. Proposition XIV, Book I

This proposition states that ([9], pg. 467):

> Prop. XIV, Thm. 6: If several bodies revolve about a common center and the centripetal force is inversely as the square of the distance of places from the center, I say that the principal latera recta of the orbits are as the squares of the areas which the bodies describe in the same time by radii drawn to the center.

If several bodies are given revolving about a common center, then both the areal velocities h and their corresponding latera recta L, are both assumed to be given. It is on this basis that Newton proceeds to establish a relationship regarding latera recta. However, notwithstanding the lack of proof for the constancy of $2h^2/L$ for several orbits, in this Proposition it is assumed that the centripetal force is varying *only* with respect to SP^2, as a *hypothesis*:

> But the minimally small line QR is in a given time as the generating centripetal force, that is (by hypothesis), inversely as SP^2.

This means that, by assuming $2h^2/L$ to be a constant, one form of Kepler's Third Law is set up as a *hypothesis* in Proposition XIV. If the bodies are moving in ellipses for example, for each individual body, Proposition XI holds. But that does not mean that $2h^2/L$ is the same for each one. There is no prior mathematical or physical requirement that ensures that the hypothesis is true, except for the actual empirical data organized by Kepler. Further in this Proposition, it is shown that:

$$h \propto \sqrt{L} \tag{13}$$

However, this is just a restatement of $2h^2/L$ being a constant, something *that holds true for one orbit alone*, generalized for "several bodies revolving around a center." Hence, the actual accomplishment of this Proposition is to introduce a form of Kepler's third law into the development of the argument, without however making it explicit. Kepler's law is implicitly *assumed* – not proved or justified. The corollary to Proposition XIV adds nothing new to the proposition, but merely restates equation (13) as:

$$A \propto \sqrt{L} \cdot T \tag{14}$$

This is done by relating areal velocity to A/T, where A is the area of the ellipse and T its periodic time.

4. Proposition XV-XVI

The previous section gives a different view of Proposition XV, which is where Kepler's Law is written out for ellipses in the *Principia*. In Proposition XV, the relationships of Proposition XIV are developed further, to show that:

$$A \propto a \cdot b \propto a\sqrt{aL} \tag{15}$$

This is from the definition of L. Combining equation (15) with (14) results in:

$$a\sqrt{a} \propto T \text{ or } a^3 \propto T^2 \tag{16}$$

However, this is simply a restatement of equation (13) in the exact same way equation (12) was obtained through an assumption of Kepler's third law. Therefore, one simply recovers from the equations what had initially been introduced into them, and the sequence from equation (10) to (16) is equivalent to a tautology: Kepler's third law implies Kepler's third law. Further proof of this deduction of an inverse-square law *from* Kepler's laws, rather than the other way around, is even found in Newton's letters ([11], pg. 143):

> … from Kepler's rule of the periodical times of the planets being in a sesquialternate proportion of their distances from the centres of their orbs, I deduced that the forces which keep the planets in their orbs must be reciprocally as the squares of the distances from the centres about which they revolve…

If indeed Kepler's laws are the source of the Propositions XIV and XV, it is interesting to note that the *Principia* does not mention Kepler even once in Book I. If the above passage is indeed accurate, the implicit application of Kepler's third law must be more closely examined.

Proposition XV says ([9], pg. 468):

> **Prop. XV Thm. 7**: Under the same suppositions as in prop. 14, I say that the squares of the periodic times in ellipses are as the cubes of the major axes.

However, Kepler's third law is originally stated thus [12]:

> But it is absolutely certain and exact that the *proportion between the periodic times of any two planets is precisely the sesquialternate proportion of their mean distances, that is, of their actual spheres,* though with this in mind, that *the arithmetic mean between the two diameters of the elliptical orbit is a little less than the longer diameter*.

There is a clear discrepancy between "… of the *major axes*" of Newton and "of their [arithmetic] *mean* distances" derived by Kepler. Thus, from equation (12) onwards, the form of Kepler's Law being implicitly assumed by Newton is different from the actual empirical law. Even if one grants that, as described by him above in his letter and as has been shown in the course of the derivations, Newton assumes Kepler's Law and derives the inverse-square law from it, it does not suffice to explain why the mean distance was replaced by the major axis, since that is not the correct form of the third law. In other words, extending equation (12), it can be said that:

$$\frac{2h^2}{L} = \frac{2(2\pi ab/T)^2}{2b^2/a} \propto \frac{a^3}{T^2} \neq \frac{(\frac{a+b}{2})^3}{T^2} \tag{12a}$$

Hence, Propositions XIV and XV are both compromised due to this approximation, as it turns out that the constant of proportionality between orbits $(2h^2/L)$ is no longer a constant – only an approximation that holds where $a \approx b$. Hence the inverse-square law results from a distortion of Kepler's law that excludes the importance of the minor axis of an ellipse, which is one of the measures of its "size". In effect, a relationship that holds true for circles is extrapolated to ellipses.

This discrepancy has immediate consequences in the corollary. The corollary to Proposition XV states that:

> Therefore the periodic times in ellipses are the same as in circles whose diameters are equal to the major axes of the ellipses.

Hence, one has the surprising result that there is no dependence of the periodic times on the minor axis of the ellipse, making a circle and an ellipse equivalent as long as the major axis matches up with the diameter. Just as in Proposition X, the implication of this corollary has also been noticed in Densmore's analysis ([10], pg. 266-267):

> It was intriguing, in fact downright strange, to see in Proposition I.10 Corollary 2 that you could squash the sides of the ellipse all you want and yet the periodic time would remain the same. The different ellipses could have different major axes as well as different minor axes…
>
> But notice that here in Proposition 15 Corollary we have something else even more astonishing. What we see is that this apparent indifference to stretching doesn't just hold for ellipses with center of forces at the center. According to this proposition and

corollary, it also holds when the center of forces is at a focus and we have an inverse-square law.

Once it is understood that the dependence on major axis has been introduced by fiat, through an altered form of Kepler's third law that applies for circles only, it becomes clear that the derivation is not just "intriguing", "strange" or "astonishing", but erroneous, and the strangeness to physical intuition follows from this fact.

Proposition XVI proceeds to establish a relationship between the velocity and the perpendicular distance between the velocity vector and the radius vector from the center of forces. Once more, it is a different rearrangement of equation (13):

$$h \propto \sqrt{L} \quad \rightarrow \quad SY \cdot |\vec{v}| \propto \sqrt{L} \quad \rightarrow \quad SY \propto \frac{\sqrt{L}}{|\vec{v}|} \tag{13a}$$

Here SY is the perpendicular let fall upon the velocity vector $\vec{v}$ (see Fig. 3). Corollaries to Proposition XVI serve either the purpose of algebraic rearrangements (Corollaries 1, 2 and 5), comparison between generic conic sections (Corollaries 6 and 7) and comparison between circles and generic conic sections (Corollaries 3, 4, 8, 9). Of these, the comparison with the circle makes use of the equivalence of major axis with radius that was already examined in the corollary of Proposition XV, and hence the same error is propagated.

Discussion

Propositions XIV, XV and XVI hence reckon with a version of Kepler's Law that is modified from the original empirical law, and it is quite possible to miss the mathematical difference between equations (12) and (12a) when it is compared with real astronomical data because in reality, eccentricities are small and the condition $a \approx b$ holds. It is hence something relatively easy to miss. The actual equation, if the correct dependence is taken into account is:

$$F = \frac{2h^2}{L \cdot r^2} = \frac{4\pi^2}{r^2}\frac{a^3}{T^2} = \frac{4\pi^2}{r^2}\frac{(\frac{a+b}{2})^3}{T^2}\left(\frac{2a}{a+b}\right)^3 = \frac{K_c}{r^2}\left(\frac{2a}{a+b}\right)^3 \tag{17}$$

Here, K_c is Kepler's constant, derived empirically, a and b are the major and minor axes of an ellipse respectively (restricting analysis to only ellipses for this paper). Since $2a/(a+b)^3$ is *not* a constant across different ellipses, the inverse-square law is an approximation at best, for ellipses of small eccentricity. The other alternative is to suggest that each orbit has its own particular inverse-square law – an approach that compromises the generality of a 'law'.

It is worth examining how the successors of Newtonian theory treated these constants. Euler, in his treatise *Mechanica* where he derives analytically several propositions from the *Principia*, says [13]:

718. Let the given curve again be an ellipse, but with the centre of force *C* placed in the other focus. The transverse axis of this is put equal to *A* and the latus rectum equal to *L*, and from the nature of the ellipse we find that $4p^2 = \frac{ALy}{A-y}$ Hence of differentiating, we have $8pdp = \frac{A^2Ldy}{(A-y)^2}$. Truly since $16p^4 = \frac{A^2L^2y^2}{(A-y)^2}$ then $\frac{dp}{2p^3} = \frac{dy}{Ly^2}$ and consequently:

$$P = \frac{4ch^2}{Ly^2}$$

Therefore the centre of force is inversely proportional to the square of the distance of the body from the centre of force C.

Here P is the force. The book carries no further treatment about whether or not $4ch^2/L$ is a constant across different conic sections. Laplace [14] describes the same process by setting up a differential equation for identifying the general force law for a general orbit, and showing that for conic sections the force law takes the form (a full quote is included here since this is the most common approach in modern notation):

[380] $$\varphi = \frac{c^2}{a\cdot(1-e^2)} \cdot \frac{1}{r^2}$$

[380'] therefore, the orbits of the planets and comets being conic sections, the force φ will be inversely proportional to the square of the distance of the centre of the planet from the centre of the sun.

We also perceive, that if the force φ be inversely proportional to the square of the distance, or expressed by h/r^2, h being a constant coefficient, the preceding equation of the conic sections, will satisfy the differential equation (4) [377] between r and v, which gives the, expression of the force, when we change φ into h/r^2. We shall then have

[380''] $$h = \frac{c^2}{a\cdot(1-e^2)}$$

which forms an equation of condition between the two arbitrary constant quantities a and e of the equation of conic sections…

[380iv] Hence it follows, that if the described curve be a conic section, the force will be in the inverse ratio of the square of the distance; and conversely, if the force be in the inverse ratio of the square of the distance, the described curve will be a conic section.

These equations are identical to equation (12) with c being the areal velocity. Laplace therefore does not hesitate to assume his h to be a constant, even though the relationship is between two arbitrary constants, and hence not a given across orbits. This constancy is not tested. Further on, he describes:

3. The intensity of the force φ, relative to each planet and comet, depends on the coefficient $\frac{c^2}{a\cdot(1-e^2)}$; the laws of Kepler furnish the means of determining it…

[382'] With respect to the planets, the law of Kepler, according to which the squares of the times of their revolutions, are as the cubes of the transverse axes of their ellipses, gives $T^2 = k^2 \cdot a^3$, k being the same for all the planets; therefore we shall have

[383] $$c = \frac{2\pi\sqrt{a(1-e^2)}}{k}$$

Hence Laplace, following Newton, takes Kepler's Law with respect to the "cubes of the transverse axes" *only*, and not *the mean distance* as originally described by Kepler. The law has been modified in this way even until recently (see [8], ch. 4), and is necessary for the derivation of the inverse-square law and conic sections from one another, whether through geometry or calculus. It must also be noted that Laplace later uses mean distances (in [385$^{\mathrm{vii}}$]) *without* thereby modifying the inverse-square relation – by simply modifying Kepler's law back to its original form.

It is necessary to clarify the preceding discussion in the light of the recent controversy over the derivation of the converse of the inverse-square law. Weinstock's position ([1] [2]) was that he agreed with the derivation from conic section orbit (CSOF) to inverse-square force (isf) law, and argues that Newton did not demonstrate the converse. Using his notation, he claims that (isf) ⇒(CSOF) was not convincingly demonstrated in the *Principia*. However, subsequent research (particularly [3]) has shown that even if the proof was not clearly demonstrated in the Principia, Newton had the tools to derive it correctly from other propositions. At the time of Newton, Kepler's laws for elliptic orbits were well-known, but the inverse-square law was yet to be demonstrated. Therefore, any proof of the inverse-square law had to satisfy both, in this order of priority:

(a) (CSOF) ⇒(isf)

(b) (isf) ⇒(CSOF)

While Weinstock and others, from the time of Bernoulli, have restricted their discussion to (b), the present discussion focuses exclusively on the validity of (a). Unless it is convincingly demonstrated that formulation of an inverse-square force for a series of orbits is justified, the converse, i.e. part (b) becomes unnecessary.

Conclusion

The importance of following through with the constants of proportionality, which mostly determine the "sizes" of the conics in one form or another, has been demonstrated by systematically examining the derivations in Propositions X, XI, XII, XIII, XIV, XV and XVI of the *Principia*, Book I. The results of this investigation are as follows:

(i) Proposition X: An error has been introduced in the force dependence equation as applied to multiple orbits by neglecting the constant relating to the area. Reference to Proposition IV is shown to be invalid. By correcting the equations, the time periods of ellipses on the same major axis but different minor axes are seen to be proportional to the minor axes, and not equal as claimed.

(ii) Propositions XI, XII, XIII: It is clarified that these propositions show only a conditional dependence on the inverse-square of the distance, since they assume that the relationship of two arbitrary constants $(2h^2/L)$ is also a constant – a fact that has to be proved. Assumption of this constancy is shown to be equivalent to assuming Kepler's third law, in *one* specific form with major axes only.

(iii) Proposition XIV, XV, XVI: Proposition XIV assumes the constancy of $2h^2/L$ for multiple orbits, without a proof or a demonstration of its truth, and Proposition XV rediscovers Kepler's law mentioned in (ii) after having implicitly assumed it. It is shown that the form of the Kepler's law should depend on both axes, not only on the major axis, making both the implicit assumption and its subsequent rediscovery in Proposition XV invalid. Proposition XVI rearranges the relations of the previous two, without adding anything new.

It is thus clear from (i), (ii) and (iii) that the primary purpose of these propositions viz. showing that conic section orbits imply inverse-square law – (CSOF) ⇒(isf) – is not satisfied, and is inconsistent with Kepler's third law. It is also seen that the same error has been repeated unacknowledged in all of the major works on the subject. The development in this paper shows that this series of propositions have to be rewritten more accurately to clarify the real principles of celestial movements.

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